SIERPIŃSKI RANK OF THE SYMMETRIC INVERSE SEMIGROUP

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Abstract. We show that every countable set of partial bijections from an infinite set to itself can be obtained as a composition of just two such partial bijections. This strengthens a result by Higgins, Howie, Mitchell and Ruškuc stating that every such countable set of partial bijections may be obtained as the composition of two partial bijections and their inverses.

1. Introduction

Cayley's Theorem states that every group is isomorphic to a subgroup of the symmetric group $\mathrm{Sym}(\Omega)$ of all permutations of some set Ω. In fact, any group G embeds in $\mathrm{Sym}(G)$.

In this sense, the semigroup-theoretic analogue of $\mathrm{Sym}(\Omega)$ is Ω^Ω, the semigroup of all functions from Ω to Ω. Every semigroup S is isomorphic to a subsemigroup of Ω^Ω for some set Ω with $|\Omega| \leq |S| + 1$.

For inverse semigroups, the corresponding object is the inverse semigroup I_Ω of all partial bijections on Ω, i.e. bijections with range and domain a subset of Ω. Every inverse semigroup S embeds into I_Ω for some set Ω with $|\Omega| \leq |S| + 1$.

The following theorem is a classical result by Sierpiński.

Theorem 1.1 ([4], Théorème I)**.** *Let Ω be an infinite set. Then every countable subset of Ω^Ω is contained in a 2-generated subsemigroup of Ω^Ω.*

Because of the property of Ω^Ω mentioned above, Theorem 1.1 immediately implies that every countable semigroup embeds in a 2-generated semigroup. In light of Theorem 1.1, the *Sierpiński rank* of a semigroup S is defined to be the least number n such that every countable subset of S is contained in an n-generated subsemigroup of S. If no such n exists, S is said to have infinite Sierpińksi rank. Note that for countable semigroups, the Sierpińksi rank is just the usual rank of a semigroup, i.e. the least size of a generating set. It was shown in [3, Lemma 2.2] that the only semigroups of Sierpiński rank 1 are 1-generated semigroups. So Theorem 1.1 says that Ω^Ω has Sierpiński rank 2.

Sierpiński ranks of various uncountable semigroups have been calculated; see the introduction of [3] for a recent survey. The following analogues of Theorem 1.1 for groups and inverse semigroups were proved by Galvin and Higgins, Howie, Mitchell and Ruškuc, respectively.

Theorem 1.2 ([1], Theorem 3.3)**.** *Let Ω be an infinite set. Then every countable subset of $\mathrm{Sym}(\Omega)$ is contained in a 2-generated subgroup of $\mathrm{Sym}(\Omega)$.*

Theorem 1.3 ([2], Proposition 4.2)**.** *Let Ω be an infinite set. Then every countable subset of I_Ω is contained in a 2-generated inverse subsemigroup of I_Ω.*

It follows from Theorem 1.2 (1.3) that every countable group (inverse semigroup) embeds in a 2-generated group (inverse semigroup).

The definition of Sierpiński rank for semigroups extends naturally to general algebras: an algebra A has Sierpiński rank n if every countable subset of A is contained in an n-generated subalgebra of A. It is easy to see that groups and inverse semigroups of Sierpiński rank 1 are commutative. So one way of stating Theorems 1.2 and 1.3 is to say that the group $\mathrm{Sym}(\Omega)$ and the inverse semigroup I_Ω have Sierpiński rank 2.

Note that the Sierpiński rank of a given object now depends on the type of algebra we choose to view it as. For instance, the Sierpiński rank of an inverse semigroup S may not be the same as the Sierpiński rank of S seen as an ordinary semigroup that "just happens to be" an inverse

semigroup. The difference is, of course, that the inverse semigroup generated by some elements of S is the semigroup generated by those elements and their inverses. So the best we can say in general is that the Sierpiński rank of the inverse semigroup S is at most the Sierpiński rank of the semigroup S which, in turn, is at most twice the Sierpiński rank of the inverse semigroup S.

There are no such difficulties between groups and inverse semigroups. The Sierpiński rank of a non-trivial group G is the same as the Sierpiński rank of the inverse semigroup G. The trivial group has Sierpiński rank 0 as a group and Sierpiński rank 1 as an (inverse) semigroup.

Since $\mathrm{Sym}(\Omega)$ and I_Ω are also important and interesting examples in the context of ordinary semigroups, it is natural to ask what their Sierpiński ranks are when seen as semigroups. In the case of $\mathrm{Sym}(\Omega)$ the answer is already known. In [1, Theorem 3.5] Galvin showed that the two generators from Theorem 1.2 may be taken to have orders 4 and 53. In particular, since they have finite orders, the semigroup generated by them is the same as the group generated by them. Hence, seen as a semigroup, $\mathrm{Sym}(\Omega)$ has Sierpiński rank 2, also.

The purpose of this short note is to prove that the Sierpiński rank of the semigroup I_Ω is also 2. In other words, to prove the following stronger version of Theorem 1.3.

Theorem 1.4. *Let Ω be an infinite set. Then every countable subset of I_Ω is contained in a 2-generated subsemigroup of I_Ω.*

2. Proof of Theorem 1.4

To prove Theorem 1.4 we require a number of preliminary results. Throughout, we will write $(x)f$ or simply xf for the image of the point x under the function f and compose functions from left to right.

Lemma 2.1. *Let $f, g \in \mathrm{I}_\Omega$ such that $\Omega g = \Omega f^{-1} = \Omega$ and $|\Omega \setminus \Omega f| = |\Omega \setminus \Omega g^{-1}| = |\Omega|$. Then for every $h \in \mathrm{I}_\Omega$ there exists $a \in \mathrm{Sym}(\Omega)$ such that $h = fag$.*

Proof. The map $f^{-1}hg^{-1} : \Omega h^{-1} f \longrightarrow \Omega hg^{-1}$ is a bijection. Since

$$|\Omega| \geq |\Omega \setminus \Omega h^{-1} f| \geq |\Omega \setminus \Omega f| = |\Omega| = |\Omega \setminus \Omega g^{-1}| \leq |\Omega \setminus \Omega hg^{-1}| \leq |\Omega|,$$

we may extend $f^{-1}hg^{-1}$ to $a \in \mathrm{Sym}(\Omega)$. Then $fag = f(f^{-1}hg^{-1})g = h$, as required. □

As mentioned earlier, the next result is an immediate consequence of Theorem 1.3. Alternatively, as shown here, it is also a corollary of Theorem 1.2.

Corollary 2.2. *Let Ω be an infinite set. Then every countable subset of I_Ω is contained in a 4-generated subsemigroup of I_Ω.*

Proof. Let A be an arbitrary countable subset of I_Ω. Let $f, g \in \mathrm{I}_\Omega$ satisfy the conditions of Lemma 2.1. Then, by Lemma 2.1, there exists a countable subset B of $\mathrm{Sym}(\Omega)$ such that $A \subseteq \langle f, g, B\rangle$. Since $\mathrm{Sym}(\Omega)$ as a semigroup has Sierpiński rank 2, there exist $h, k \in \mathrm{Sym}(\Omega)$ such that $B \subseteq \langle h, k\rangle$. Thus $A \subseteq \langle f, g, B\rangle \subseteq \langle f, g, h, k\rangle$, as required. □

Recall that an element i of $\mathrm{Sym}(\Omega)$ is called an *involution* if i^2 equals the identity 1_Ω on Ω. The following is a well-known result, see, for example, [1, Lemma 2.2].

Lemma 2.3. *Every element of $\mathrm{Sym}(\Omega)$ is a product of two involutions.*

We will also require the following result, the proof of which is similar to that of Lemma 2.3.

Lemma 2.4. *For every $a \in \mathrm{Sym}(\Omega)$ there exists an involution $j \in \mathrm{Sym}(\Omega)$ such that $a^{-1} \in \langle a, aj\rangle$.*

Proof. Let σ be any cycle of a and fix an arbitrary x in the orbit of σ. Define the transformation j_σ of the orbit $\{x\sigma^n : n \in \mathbb{Z}\}$ of σ by $(x\sigma^n)j_\sigma = x\sigma^{-n+1}$ for all $n \in \mathbb{Z}$. Note that $(x\sigma^{-n+1})j_\sigma = x\sigma^{-(-n+1)+1} = x\sigma^n$ and so j_σ is an involution on the orbit of σ.

Furthermore, $(x\sigma^n)\sigma j_\sigma = (x\sigma^{n+1})j_\sigma = x\sigma^{-n}$ and so

$$(x\sigma^n)\sigma j_\sigma \sigma\sigma j_\sigma = (x\sigma^{-n})\sigma\sigma j_\sigma = (x\sigma^{-n+1})\sigma j_\sigma = x\sigma^{n-1}.$$

Thus $(\sigma j_\sigma)\sigma(\sigma j_\sigma) = \sigma^{-1}$.

In the same way as above, define j_τ for every cycle τ of a and let j be the union of all j_τ. Then $j \in \mathrm{Sym}(\Omega)$ is an involution and $(aj)a(aj) = a^{-1}$. In particular, $a^{-1} \in \langle a, aj\rangle$, as required. □

We are now in a position to prove the main theorem.

Proof of Theorem 1.4. By Corollary 2.2, it suffices to show that for all $h_1, h_2, h_3, h_4 \in \mathrm{I}_\Omega$ there exist $f, g \in \mathrm{I}_\Omega$ such that $h_1, h_2, h_3, h_4 \in \langle f, g\rangle$. Partition Ω into countably infinitely many sets $\Omega_0, \Omega_1, \Omega_2 \ldots$ where $|\Omega_i| = |\Omega|$ for every $i \in \mathbb{N}$. Let f be any element of I_Ω that maps Ω_i bijectively to Ω_{i+1} for every $i \in \mathbb{N}$. Note that $|\Omega \setminus \Omega f| = |\Omega_0| = |\Omega|$ and $\Omega f^{-1} = \Omega$.

For $13 \leq n \leq 22$, let $i_n \in \mathrm{Sym}(\Omega_n)$ be an involution and let g be any element of I_Ω with domain $\bigcup_{n=13}^{\infty} \Omega_n$ such that:

- $g|_{\Omega_n} = i_n$ for $13 \leq n \leq 22$;
- $(\Omega_{23})g = \Omega_{23} \cup \Omega_{24}$;
- $(\Omega_{24})g = \bigcup_{n=25}^{\infty} \Omega_n$;
- $(\Omega_{25})g = \bigcup_{n=1}^{12} \Omega_n$;
- $(\bigcup_{n=26}^{\infty})g = \Omega_0$.

The aim is now to specify the involutions i_n in such a way that $h_1, h_2, h_3, h_4 \in \langle f, g\rangle$. The definition of i_n will depend on h_1, h_2, h_3, h_4, f and g. Since g, in turn, depends on the i_n, we must be very careful to avoid circular definitions.

Note that g^2 is independent of the choices for the i_n (as long as every i_n is indeed an involution). Note that the domain of g^2 is $(\Omega)g^{-2} = (\bigcup_{n=13}^{\infty})g^{-1} = \bigcup_{n=13}^{24} \Omega_n$ and the range is $\Omega g^2 = \Omega g = \Omega$. Let $\pi = f^{26}g$ and $\tau = g^{-2}f^{-12}g^{-1}f^{-25}$. It is easy to verify that π and τ are bijections from Ω to Ω_0. Furthermore, π is independent of the choices for the i_n, since $\Omega f^{26} = \bigcup_{n=26}^{\infty}$ has empty intersection with the union $\bigcup_{n=13}^{22} \Omega_n$ of the domains of the i_n. Similarly, τ is independent of the choices for i_n, since g^2, and hence g^{-2}, are independent and $\Omega g^{-2} f^{-12} = (\bigcup_{n=13}^{24} \Omega_n)f^{-12} = \bigcup_{n=1}^{12} \Omega_n$ has empty intersection with the union $\bigcup_{n=13}^{22} \Omega_n$ of the domains of the i_n^{-1}. In particular, we may, without fear of our argument becoming circular, use g^2, π and τ when defining i_n.

Since f and g^2 satisfy the conditions of Lemma 2.1, there exist $a_1, a_2, a_3, a_4 \in \mathrm{Sym}(\Omega)$ such that $h_1, h_2, h_3, h_4 \in \langle f, g^2, a_1, a_2, a_3, a_4\rangle$. By Lemma 2.3, there exist involutions $j_1, \ldots, j_8 \in \mathrm{Sym}(\Omega)$ such that $a_1, a_2, a_3, a_4 \in \langle j_1, \ldots, j_8\rangle$. Then $h_1, h_2, h_3, h_4 \in \langle f, g^2, j_1, \ldots, j_8\rangle$. Since π and τ are both bijections from Ω to Ω_0, the composite $\pi\tau^{-1}$ is an element of $\mathrm{Sym}(\Omega)$. Hence, by Lemma 2.4, there exists an involution $j_9 \in \mathrm{Sym}(\Omega)$ such that $(\pi\tau^{-1})^{-1} \in \langle(\pi\tau^{-1}), (\pi\tau^{-1})j_9\rangle$. Let j_{10} be the identity Ω.

Note that τf^n is a bijection from Ω to Ω_n and define

$$i_n = (\tau f^n)^{-1} j_{n-12} (\tau f^n) = (f^{-n}\tau^{-1} j_{n-12} \tau f^n)|_{\Omega_n}$$

for $13 \leq n \leq 22$. Then i_n is an involution since it is the conjugate of the involution j_{n-12}. Furthermore, if $x \in \Omega$ is arbitrary, and $1 \leq k \leq 10$, then $(x)f^{26}gf^{12+k} = (x)\pi f^{12+k} \in \Omega_{12+k}$. Hence

$$\begin{aligned}
(x)f^{26}gf^{12+k}gf^{13-k}gf^{12}g^2 &= (x)\pi f^{12+k}(f^{-12-n}\tau^{-1}j_k\tau f^{12+k})f^{13-k}gf^{12}g^2\\
&= (x)\pi\tau^{-1}j_k\tau f^{25}gf^{12}g^2\\
&= (x)\pi\tau^{-1}j_k\tau\tau^{-1}\\
&= (x)\pi\tau^{-1}j_k.
\end{aligned}$$

Thus $f^{26}gf^{12+k}gf^{13-k}gf^{12}g^2 = (\pi\tau^{-1})j_k$. In particular, $(\pi\tau^{-1})j_k \in \langle f, g\rangle$ for $1 \leq k \leq 10$. But j_9 was chosen such that $(\pi\tau^{-1})^{-1} \in \langle(\pi\tau^{-1}), (\pi\tau^{-1})j_9\rangle$ and $(\pi\tau^{-1}) = (\pi\tau^{-1})j_{10}$. Hence $(\pi\tau^{-1}) \in \langle f, g\rangle$. It follows that $j_1, \ldots, j_8 \in \langle f, g\rangle$. Thus

$$h_1, h_2, h_3, h_4 \in \langle f, g^2, j_1, \ldots, j_8\rangle \subseteq \langle f, g\rangle,$$

as required. □

References

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